

EXPECTED EULER CHARACTERISTIC OF EXCURSION SETS OF RANDOM HOLOMORPHIC SECTIONS ON COMPLEX MANIFOLDS

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ABSTRACT. We prove a formula for the expected euler characteristic of excursion sets of random sections of powers of an ample bundle (L, h) , where h is a Hermitian metric, over a Kähler manifold (M, ω) . We then prove that the critical radius of the Kodaira embedding $\Phi_N : M \rightarrow \mathbb{CP}^n$ given by an orthonormal basis of $H^0(M, L^N)$ is bounded below when $N \rightarrow \infty$. This result also gives conditions about when the preceding formula is valid.

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1. INTRODUCTION

Let M be a Kähler manifold of dimension m . And let $L \rightarrow M$ be an ample line bundle with positively curved metric h . Take the induced Kähler form $\omega = \frac{i}{2}\Theta_h$ on M . We denote by L^N the N th tensor power $L^{\otimes N}$ of L . Take the induced metric on L^N , by abuse of notation, also denoted by h . This induces a Hermitian inner product in $H^0(M, L^N)$, given by

$$\langle \sigma_1, \sigma_2 \rangle = \frac{1}{m!} \int_M h(\sigma_1, \sigma_2) \omega^m$$

. In particular, the L^2 norm of a section in $s \in H^0(M, L^N)$ is given by

$$|s|_h^2 = \frac{1}{m!} \int_M |s(z)|_h^2 \omega^m$$

We consider random sections in the unit sphere S_L^N in $H^0(M, L^N)$ with probability measure given by the spherical volume normalized so that $\text{Vol}(S_L^N) = 1$. For $s \in S_L^N$, the zero locus $Z_s = \{z \in M | s(z) = 0\}$ is very well studied in [SZ1][SZ3][SZ2][SZ5]. It is also interesting to understand the excursion sets $\{z \in M | |s(z)|_h > u\}$. In particular, what is $E\chi(|s(z)|_h > u) = \int_{S_L^N} \chi(|s(z)|_h > u) ds$, the expected Euler characteristic of the excursion

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sets, and what is the probability that the excursion set is non-empty. Here and in the following we denote by $\chi(S)$ the Euler characteristic of a topological space S .

It turned out that it is more natural to normalize the excursion sets to be of the form $\{\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > u\}$, where $\Pi_N(z,z)$, which is in general not constant but of the form $\frac{N^m}{\pi^m}(1 + O(\frac{1}{N}))([Ca][Ti][Z])$, is the Szegő kernel of $H^0(M, L^N)$. By the definition of $\Pi_N(z,z)$, we always have $\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} \leq 1$. In fact $\sup_{|s|_h=1} |s(z)|_h^2 = \Pi_N(z,z)([B])$. Therefore when $u > 1$, the excursion sets are empty.

In this paper, we will mainly prove two theorems.

The first theorem is interesting in itself. Also it shows that in order to have a nice formula for the expected Euler characteristic we do not need to make u too close to 1.

Theorem 1.1. *Let $\Phi_N : M \rightarrow \mathbb{CP}^n$ be an embedding given by an orthonormal basis of $H^0(M, L^N)$. Let r_N be the critical radius of $\Phi_N(M)$ considered as a submanifold of \mathbb{CP}^n . Then there exists a constant $\rho_0(L, h) > 0$ such that $r_N > \rho_0(L, h)$ for all positive integer N .*

The proof of this theorem depends mainly on the approximation of the normalized Szegő kernel defined and proved in [SZ1]. The idea is based on the sense that the information of the embedding $\Phi_N : M \rightarrow \mathbb{CP}^n$ is totally contained in the normalized Szegő kernel.

The second one is to answer the question about expected Euler characteristic of the normalized excursion set.

Theorem 1.2. *Let $\Phi_N : M \rightarrow \mathbb{CP}^n$ be an embedding given by an orthonormal basis of $H^0(M, L^N)$. Then there exists $\rho_0 > 0$ independent of N , such that for $0 \leq \rho < \rho_0$, the set $\{z \in M \mid \frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos \rho\}$ is either empty or contractible, therefore*

$$E\chi\left(\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos \rho\right) = \text{Prob.}\left\{\sup_{z \in M} \frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos \rho\right\}$$

Hence the following formula

$$E\chi\left(\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > \cos \rho\right) = \int_M c(M)(1 - Nc_1(L)) \wedge (Nc_1(L) \cos^2 \rho + \sin^2 \rho)^n \quad (1)$$

Where $c_1(L)$ is the first Chern class of L and $c(M)(1 - Nc_1(L))$ is the Chern polynomial evaluated at $1 - Nc_1(L)$

When M is a Riemann surface, we have a more explicit formula

Theorem 1.3. *Let M be a Riemann surface. Then, with the notations above, there exists $\rho_N > 0$ such that for $u > \cos \rho_N$ and a random section $s(z) \in H^0(M, L^N)$ the expected Euler characteristic*

$$E\chi\left(\frac{|s(z)|_h}{\sqrt{\Pi_N(z,z)}} > u\right) = (1 - u^2)^{(n-1)}[N^2(\deg L)^2 u^2 \quad (2)$$

$$- N \deg L(gu^2 - 1 + u^2) + (2 - 2g)(1 - u^2)] \quad (3)$$

where $n = N \deg(L) - g$ for $N \deg(L) > 2g - 2$

When M is higher dimensional, we can only get an estimation

Theorem 1.4. *With the notations above, for $m \geq 1$ and for N big enough*

$$E\chi\left(\frac{s(z)}{\sqrt{\Pi_N(z, z)}} > u\right) = (1 + o(1))n^{m+1}(1 - u^2)^{n-m}u^{2m}$$

where

$$n = \dim H^0(M, L^N) - 1 = \frac{\int_M c_1^m(L)}{m!} N^m + O(N^{m-1})$$

where the second equality follows from the asymptotic Riemann-Roch formula.

Our results are complementary to results on excursion probabilities for Gaussian fields (see [AT][SU]) where the probability of large L^2 norms plays a role. Here, we consider only sections with L^2 norm 1.

Notice that by our estimation $E\chi\left(\frac{s(z)}{\sqrt{\Pi_N(z, z)}} > u\right)$ decays to 0 very rapidly (exponentially) as N goes to ∞ . It is helpful to compare this observation with the following theorem from [SZ4], which we state using our notations

Theorem 1.5. *(Theorem 1.1, [SZ4]) Let ν_N denote the measure on S_L^N induced by the metric ds , for any integer k , there exist constants $C > 0$ depending on k , such that*

$$\nu_N\{s_N \in S_L^N : \sup_{z \in M} |s_N(z)|_h > C\sqrt{\log N}\} < O\left(\frac{1}{N^k}\right)$$

Normalizing the above formula using $\sqrt{\Pi_N(z, z)}$, and by the estimation of $\sqrt{\Pi_N(z, z)}$, we have

$$\nu_N\{s_N \in S_L^N : \sup_{z \in M} \frac{|s_N(z)|_h}{\sqrt{\Pi_N(z, z)}} > \frac{C\sqrt{\log N}}{N^{m/2}}\} < O\left(\frac{1}{N^k}\right)$$

the term $\frac{C\sqrt{\log N}}{N^{m/2}}$ is very small when N is big. But the estimation we made requires u close to 1, so in this sense our estimation is weaker, although it is more explicit.

It should be mentioned here that in the proof of theorem 1.2, we make ρ_0 small enough so that the Euler characteristic of an excursion set is either 1 or 0. The author does not have an idea on the general situation when ρ_0 is greater than the critical radius (section 4). It seems that we need to understand the exterior geometry of the embedded manifolds better. It is an interesting question to ask, for example, when does an excursion set have 2 components? More generally, what is the volume of the set of sections whose excursion sets have k components.

The paper is organized as follows: first in section 2 we introduce the definition of Szegő Kernel and state several results from [SZ3], [SZ1], and [Z]. In section 3 we prove the formula in theorem 1.2. In section 4 we will analyze the critical radius, by using the results stated in section 2 first in the case of Riemann surfaces then generalize to high dimensional case.

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2. SZEGÖ KERNEL

We will follow the notations and arguments in [SZ1] and [SZ3]

Let $L \rightarrow M$ be a positive line bundle over a Kähler manifold M . The associated principle sphere bundle is defined as follows. Let $\pi : L^* \rightarrow M$ be the dual bundle to L with dual metric h^* . And put $X = \{v \in L^* : \|v\|_h^* = 1\}$. Let $r_\theta x = e^{i\theta}x$ ($x \in X$) denote the S^1 action on X . Now any section $s \in H^0(M, L^N)$ is lifted to an equivariant function \hat{s} on the circle bundle $\pi : X \rightarrow M$ with respect to h by the rule

$$\hat{s}(\lambda) = (\lambda^{\otimes N}, s(z)), \lambda \in X_z$$

where $\lambda^{\otimes N} = \lambda \otimes \cdots \otimes \lambda$. Let $(s_j^N) \subset H^0(M, L^N)$ be an orthonormal basis. The Szegö kernel is given by

$$\Pi_N(x, y) = \sum_{i=0}^n \hat{s}_j^N(x) \hat{s}_j^N(y) \quad (x, y \in X)$$

The normalized Szegö kernel is defined as

$$P_N(z, w) := \frac{|\Pi_N(z, w)|}{\Pi_N(z, z)^{1/2} \Pi_N(w, w)^{1/2}}$$

where

$$|\Pi_N(z, w)| := |\Pi_N(x, y)|, \quad z = \pi(x), w = \pi(y) \in M. \quad (4)$$

On the diagonal we have

$$\Pi_N(z, z) = \sum_{i=0}^n \|s_j^N(z)\|_h^2, z \in M$$

The following theorem was proved in [Z]

Theorem 2.1 ([Z]). *Let M be a compact complex manifold of dimension m (over \mathbb{C}) and let $(L, h) \rightarrow M$ be a positive hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g := \text{Ric}(h)$. For each $N \in \mathbb{N}$, h induces a hermitian metric h_N on $L^{\otimes N}$. Let $\{S_0^N, \dots, S_{d_N}^N\}$ be any orthonormal basis of $H^0(M, L^{\otimes N})$, with respect to the inner product $\langle s_1, s_2 \rangle_{h_N} = \int_M h_N(s_1(z), s_2(z)) dV_g$. Here, $dV_g = \frac{1}{m!} \omega_g^m$ is the volume form of g . Then there exists a complete asymptotic expansion:*

$$\sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 \sim a_0 N^m + a_1(z) N^{m-1} + \dots$$

for certain smooth coefficients $a_j(z)$ with $a_0 = \frac{1}{\pi^m}$. More precisely, for any k

$$\left| \sum_{i=0}^{d_N} \|S_i^N(z)\|_{h_N}^2 - \sum_{j < R} a_j(z) N^{m-j} \right|_{C^k} \leq C_{R,k} N^{m-R}$$

At a point $z_0 \in M$, we choose a neighborhood U of z_0 , a local normal coordinate chart $\rho : U, z_0 \rightarrow \mathbb{C}^m$, 0 centered at z_0 , and a preferred local frame at z_0 , which was defined in [SZ3] to be a local frame e_L such that

$$|e_L(z)|_h = 1 - 1/2 |\rho(z)|^2 + \dots$$

The following theorem was proved in [SZ1]

Theorem 2.2 ([SZ1], Proposition 2.7). *Let $(L, h) \rightarrow (M, \omega)$ be a positive Hermitian holomorphic line bundle over a compact m -dimensional Kähler manifold M . We give $H^0(M, L^N)$ the Hermitian Gaussian measure induced by h and the Kähler form $\omega = \frac{i}{2}\Theta_h$. And let $P_N(z, w)$ be the normalized Szegő kernel for $H^0(M, L^N)$ and let $z_0 \in M$. For $b, \varepsilon > 0, j \geq 0$, there is a constant $C_j = C_j(M, \varepsilon, b)$, independent of the point z_0 , such that*

$$P_N(z_0 + \frac{u}{\sqrt{N}}, z_0 + \frac{v}{\sqrt{N}}) = e^{-\frac{1}{2}|u-v|^2} [1 + R_N(u, v)] \quad (5)$$

$$|\nabla^j R_N(u, v)| \leq C_j N^{-1/2+\varepsilon} \quad \text{for } |u| + |v| < b\sqrt{\log N} \quad (6)$$

As a corollary we have

Theorem 2.3 ([SZ1], Proposition 2.8). *The remainder R_N in the above theorem satisfies*

$$|R_N(u, v)| \leq \frac{C_2}{2} |u - v|^2 N^{-\frac{1}{2}+\varepsilon}, \quad |\nabla R_N(u, v)| \leq C_2 |u - v| N^{-\frac{1}{2}+\varepsilon}, \quad (7)$$

$$\text{for } |u| + |v| < b\sqrt{\log N}. \quad (8)$$

3. EXPECTED EULER CHARACTERISTIC

Let $s_N^j \in H^0(M, L^N), 0 \leq j \leq n$, where $n + 1 = \dim(H^0(M, L^N))$ be an orthonormal basis. By the Kodaira embedding theorem, for N big enough this gives an embedding $\Phi_N : M \rightarrow \mathbb{CP}^N$, locally given by $\Phi_N(x) = [f_N^0(x), f_N^1(x), \dots, f_N^n(x)]$, where $s_N^j = f_N^j e_L^N$ with e_L a local frame of L . For a random section with norm 1,

$$s = \sum_{i=0}^N c_i s_N^i, \quad \sum_{i=0}^N \|c_i\|^2 = 1$$

Let $|s(z)|_h$ denote the norm of s at $z \in M$ under the metric induced by h . Let $C = (c_i)$ and $f_N(z) = (f_N^i(z))$. Then $\sum_{i=0}^N |s_N^i(z)|_h^2 = |f_N(z)|^2 |e_L(z)|_h^{2N}$

$$|s(z)|_h = |C \cdot f_N(z)| |e_L|^N = \frac{C \cdot f_N(z)}{|C| |f_N(z)|} \sqrt{\sum_{i=0}^N |s_N^i(z)|_h^2}$$

By definition $\Pi_N(z, z) = \sum_{i=0}^n |s_N^i(z)|_h^2$ is just the Szegő kernel for $H^0(M, L^N)$ on the diagonal.

Therefore we have

$$\frac{|s(z)|_h}{\sqrt{\Pi_N(z, z)}} = \frac{|C \cdot \Phi_n(z)|}{|C| |\Phi_n(z)|}$$

Notice that with the Fubini-Study metric on \mathbb{CP}^n ,

$$\cos d_{FS}(C, \Phi_N(Z)) = \frac{|C \cdot \Phi_N(Z)|}{|C| |\Phi_n(Z)|}$$

So we get

$$\frac{|s(Z)|_h}{\sqrt{\Pi_N(z, z)}} = \cos d_{FS}(C, \Phi_N(Z))$$

Lemma 3.1. *Let M be a compact submanifold of (A, g) , where (A, g) is a C^∞ Riemannian manifold, let $B_\rho(P)$ denote the ball centered at $P \in A$ with radius ρ , then for $\rho > 0$ small enough, $B_\rho(P) \cap M$ is contractible if not empty.*

Proof. Consider the normal bundle of $\pi : N \rightarrow M$ in A and the exponential map $\exp : N \rightarrow A$. Since M is compact, there exists $\rho_1 > 0$ such that restricted to the open neighborhood $O_M(\rho_1) = \{(p, v) \mid \|v\| < \rho_1\} \subset N$, the exponential map is injective. Now we claim that any $\rho < \rho_1$ satisfies the requirement of the lemma.

Now suppose $B_\rho(P) \cap M$ is not empty, then $P \in \exp(O_M(\rho_1))$ and $P = \exp(p, v)$ with $(p, v) \in O_M(\rho_1)$. Consider $d(P, -)$ as a smooth function on $B_\rho(P) \cap M$, then by assumption p is the only critical point of $d(P, -)$, since the geodesic that connects p and a critical point of $B_\rho(P) \cap M$ is orthogonal to M . Let $r = d(P, p)$, then $B_r(P) \cap M = p$. So by Morse theory $B_\rho(P) \cap M$ is contractible. \square

Corollary 3.2. *Let r_N be the critical radius of the embedding $\Phi_N(M) \subset \mathbb{CP}^n$, then for $\rho < r_N$, the excursion set $\{z \in M \mid \frac{|s(z)|_h}{\sqrt{\Pi_N(z, z)}} > \cos \rho\}$ is either contractible or empty.*

By theorem 1.1 (which will be proved in the next section), r_N is bounded below by $\rho_0 > 0$. Therefore as a corollary, taking into account that the Fubini-Study metric on \mathbb{CP}^n is the quotient of the "round metric" under the fibration $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ we have

$$E\chi\left(\frac{|s(z)|_h}{\sqrt{\Pi_N(z, z)}} > \cos \rho\right) = \text{Prob.}\left\{\sup_{z \in M} \frac{|s(z)|_h}{\sqrt{\Pi_N(z, z)}} > \cos \rho\right\} = \frac{\text{Vol}(T(\Phi_N(M), \rho))}{\text{Vol}(\mathbb{CP}^n)}$$

for $\rho < \rho_0$

First we calculate the volume $V(T(\Phi_N(M), \rho))$. We use theorems and formulas from [G2] (Theorem 7.20) (see also [G1]).

Theorem 3.3. *Let M^m be an embedded complex submanifold of $(\mathbb{CP}^n, \omega_{FS})$, and let N be the normal bundle of M in \mathbb{CP}^n suppose that $\exp : \{(p, v) \in N \mid \|v\| < r\} \rightarrow T(M, r)$ is a diffeomorphism. Then*

$$V_M(r) = \frac{1}{n!} \int_M \prod_{a=1}^m \left(1 - \frac{\omega_{FS}}{\pi} + x_a\right) \wedge (\pi \sin^2(r) + \cos^2(r) \omega_{FS})^n$$

Where x_a is defined formally in factorization of the Chern polynomial $c(M)(t) = \prod_{a=1}^m (t + x_a)$

As a corollary of this theorem and by plugging in $\Phi_N^*(\omega_{FS}) = N\pi c_1(L)$, and dividing by the Fubini-Study volume $\pi^n/n!$ of CP^n , we get theorem 1.2

When M is a Riemann surface, $m = 1$, so $x_1 = c_1(M)$ the first Chern class. So

$$V_M(r) = \frac{1}{n!} \int_M \left(1 - \frac{\omega_{FS}}{\pi} + c_1(M)\right) \wedge [(\pi \sin^2(r))^n + n(\pi \sin^2(r))^{n-1} \cos^2(r) \omega_{FS}]$$

therefore

$$V_M(r) = \frac{1}{n!} \int_M [(\pi \sin^2(r))^n (c_1(M) - \frac{\omega_{FS}}{\pi}) + n(\pi \sin^2(r))^{n-1} \cos^2(r) \omega_{FS}]$$

We know by the Gauss-Bonnet formula $\int_M c_1(M) = \chi(M) = 2 - 2g$ and since $\Phi_N(N)$ is of degree $N \deg(L)$ in \mathbb{CP}^n , $\int_M \omega_{FS} = N \deg(L) \pi$. Now we can write out the explicit formula for $V(T(\Phi_N(M), \rho))$, that is

$$\begin{aligned} V(T(\Phi_N(M), \rho)) &= \frac{1}{n!} [(\pi \sin^2(\rho))^n (\chi(M) - N \deg(L)) + nN \deg(L) \pi (\pi \sin^2(\rho))^{n-1} \cos^2(\rho)] \quad (9) \\ &= \frac{\pi^n}{n!} (\sin^{2(n-1)} \rho) [N^2 (\deg L)^2 \cos^2 \rho - N \deg L (g \cos^2 \rho - \sin^2 \rho) + (2 - 2g) \sin^2 \rho] \quad (10) \end{aligned}$$

where $\chi(M) = 2 - 2g$ and by Riemann-Roch formula $n = N \deg(L) - g$ for $N \deg(L) > 2g - 2$

To summarize, we have the following theorem

Theorem 3.4. *Let M be a Riemann surface. Then, with the notations above, there exists $\rho_0 > 0$ such that for $\rho < \rho_0$, $N \deg L > 2g - 2$ and a random section $s(z) \in H^0(M, L^N)$ the expected Euler characteristic*

$$E\chi\left(\frac{s(z)}{\sqrt{\Pi_N(z, z)}} > \cos \rho\right) \quad (11)$$

$$= (\sin^{2(n-1)} \rho) [N^2 (\deg L)^2 \cos^2 \rho - N \deg L (g \cos^2 \rho - \sin^2 \rho) + (2 - 2g) \sin^2 \rho] \quad (12)$$

If we write $u = \cos \rho$ and plug in $\sin^2 \rho = 1 - u^2$, we get theorem 1.3

Note that when $m > 1$, the expansion of $E\chi\left(\frac{s(z)}{\sqrt{\Pi_N(z, z)}} > \cos \rho\right)$ is complicated and the author can not get a more intuitive formula. We can calculate the leading term to have an estimation of formula 1.

Observe that the leading term in the expansion should be

$$\frac{1}{\pi^n} \int_M \binom{n}{m} (\pi \sin^2 \rho)^{n-m} (\cos^2 \rho \omega_{FS})^m$$

Let $O_n(1)$ denote the hyperplane bundle on \mathbb{CP}^n . Then ω_{FS} is a multiple of the first Chern class of $O_n(1)$ that is $\omega_{FS} = \pi c_1(O_n(1))$. Also the pull back $\Phi_N^*(c_1(O_n(1))) = N c_1(L)$. Therefore we have

$$\int_M \omega_{FS}^m = \pi^m N^m \int_M c_1^m(L)$$

which is independent of the metric on L .

So formula 3 becomes

$$\binom{n}{m} (\sin^2 \rho)^{n-m} (\cos^2 \rho)^m N^m \int_M c_1^m(L)$$

By the asymptotic Riemann-Roch formula (ref. Theorem 1.1.22[L]) for N big enough

$$n = \frac{\int_M c_1^m(L)}{m!} N^m + O(N^{m-1})$$

So the leading term is about

$$n^{m+1} (\sin^2 \rho)^{n-m} (\cos^2 \rho)^m$$

Therefore we have the following theorem

Theorem 3.5. *With the notations above, for $m \geq 1$ and for N big enough*

$$E\chi\left(\frac{s(z)}{\sqrt{\Pi_N(z, z)}} > \cos \rho\right) = (1 + o(1))n^{m+1}(\sin^2 \rho)^{n-m}(\cos^2 \rho)^m$$

Again, plugging in $u = \cos \rho$, we get theorem 1.4

Let $r_n = \sup\{\rho_n \text{ that satisfies the requirement of theorem 1.3}\}$. we are going to show that although r_n might get smaller as n grows, there is a positive lower bound.

4. CRITICAL RADIUS

We will first analyze the case of Riemann surfaces, then generalize the results to that of higher dimensional smooth projective variety.

First we talk about a little geometry of \mathbb{CP}^n with the Fubini-Study metric. For any $p = [c_0, c_1, \dots, c_n] \in \mathbb{CP}^n$ the points q such that $d(p, q) = \pi/2$ form a hyperplane $H(p)$ defined by $\sum_0^n c_i \bar{z}_i = 0$ and for any $q \in H$ the complex line that connects p and q is orthogonal to $H(p)$ at q . We call H the orthogonal hyperplane of q . Conversely, any complex line that are orthogonal to $H(p)$ at a point $q \in H(p)$ must goes through p . Also any linear subspace is geodesic.

4.1. Riemann Surfaces. Let X be a compact Riemann Surface, and let (L, h) be a positive Hermitian holomorphic line bundle over X . The curvature of (L, h) induces a Kähler metric on X with Kähler form $\omega = \frac{i}{2}\Theta_h$. Let s_0, s_1, \dots, s_n be an orthonormal basis of $H^0(X, L^N)$. Here we write n instead of $n(N)$ for short. This gives an embedding $\Phi_N : X \rightarrow \mathbb{CP}^n$ for N big enough by Kodaira. If we choose a holomorphic local frame e_L of L , then $s_i = f_i e_L^N$ with f_i holomorphic functions. So Φ_N is locally given by $\Phi_N(z) = [f_0(z), f_1(z), \dots, f_n(z)]$. We denote the vector $(f_0(z), f_1(z), \dots, f_n(z)) \in \mathbb{C}^{n+1}$ by $F(z)$, and the vector $(f'_0(z), f'_1(z), \dots, f'_n(z))$ by $F'(z)$. At each point $\Phi_N(z)$, the holomorphic tangent line is given by $[F(z) + tF'(z)], t \in \mathbb{C} \cup \{\infty\}$. By $[v]$ for $v \in \mathbb{C}^{n+1}$, we mean the image under the projection $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{CP}^n$. Consider the normal bundle $N \subset T\mathbb{CP}^n|_{\Phi_N(X)}$. At any point $p \in \Phi_N(X)$, $\exp(N_p)$ is the hyperplane H_p passing p which is orthogonal to $T_p\Phi_N(X)$ at p . We define $T_\infty(z)$ as the only point on the tangent line through $\Phi_N(z)$ with distance $\pi/2$ to $\Phi_N(z)$ in \mathbb{CP}^n . Then

$$[T_\infty(z)] = [F'(z) - \frac{\langle F'(z), F(z) \rangle}{\langle F(z), F(z) \rangle} F(z)]$$

We denote by $O_z()$ the projection of \mathbb{CP}^n from $T_\infty(z)$ to its orthogonal hyperplane, which is just H_z . In particular we have

$$[O_z([v])] = [v - \frac{\langle v, T_\infty(z) \rangle}{|T_\infty(z)|^2} T_\infty(z)]$$

Also by $d_N(,)$ we mean the distance in \mathbb{CP}^n induced by the Fubini-Study metric.

Lemma 4.1. *Let $H_{z \cap w}$ denote the intersection of the normal hyperplanes H_z, H_w of $\Phi_N(X)$ through $\Phi_N(z)$ and $\Phi_N(w)$ respectively, then*

$$\sin^2(d_N(\Phi_N(z), H_{z \cap w})) = \cos^2(d_N(\Phi_N(z), O_z(T_\infty(w))))$$

Proof. By unitary change of coordinates, we can assume that $\Phi_N(z) = [0, \dots, 0, 1]$, and that $T_\infty(z) = [0, \dots, 0, 1, 0]$. For any $q \in H_{z \cap w}$, let $q = [v_0, v_1, \dots, v_n]$ with $\sum_{i=0}^n |v_i|^2 = 1$. Then $\cos(d_N(\Phi_N(z), q)) = |v_n|$. So $\cos^2(d_N(\Phi_N(z), H_{z \cap w})) = \max |v_n|^2$. Let $T_\infty(w) = [c_0, c_1, \dots, c_n]$. So the v_i 's satisfies the following equations

$$v_{n-1} = 0, \quad \sum_{i=0}^n c_i \bar{v}_i = 0$$

. So the maximum $|v_n|$ is

$$|v_n|^2 = 1 - \frac{|c_n|^2}{\sum_{i \neq n-1} |c_i|^2}$$

On the other hand it is clear that

$$\cos^2(d_N(\Phi_N(z), O_z(T_\infty(w)))) = \frac{|c_n|^2}{\sum_{i \neq n-1} |c_i|^2}$$

Combining the equations, we get the conclusion. \square

Therefore, by switching z and w , we have the following equation

$$\sin^2(d_N(\Phi_N(w), H_{z \cap w})) = \frac{|\langle F(w), O_w(T_\infty(z)) \rangle|^2}{|F(w)|^2 |O_w(T_\infty(z))|^2}$$

where by abuse of notation, we consider the homogeneous coordinate of a point in \mathbb{CP}^n as a vector in \mathbb{C}^{n+1} .

Before we go on calculating the right side of the equation, we recall the normalized Szegő kernels in [SZ1] is defined as

$$P_N(z, w) := \frac{|\Pi_N(z, w)|}{\Pi_N(z, z)^{1/2} \Pi_N(w, w)^{1/2}}$$

Since $|s_i|_h^2 = |f_i|^2 h^N$, we have

$$P_N(z, w) = \frac{|\langle F(z), F(w) \rangle|}{\langle F(z), F(z) \rangle^{1/2} \langle F(w), F(w) \rangle^{1/2}}$$

Now we let $E(z, w) = P_N^2(z, w)$, then

$$E(z, w) = \frac{\langle F(z), F(w) \rangle \langle F(w), F(z) \rangle}{\langle F(z), F(z) \rangle \langle F(w), F(w) \rangle}$$

Therefore

$$\frac{\partial}{\partial z} E(z, w) = \frac{\langle F(w), F(z) \rangle}{|F(z)|^2 |F(w)|^2} [\langle F'(z), F(w) \rangle - \frac{\langle F(z), F(w) \rangle}{|F(z)|^2} \langle F'(z), F(z) \rangle]$$

and

From now on we use the following convention, by $Df(*, *)$, where D is some differential, we always mean the value of Df at $(*, *)$

$$\frac{\partial^2}{\partial z \partial \bar{w}} E(z, w) = \frac{\langle F(w), F(z) \rangle}{|F(z)|^2 |F(w)|^2} [\langle F'(z), F'(w) \rangle \quad (13)$$

$$- \frac{\langle F(w), F'(w) \rangle}{|F(w)|^2} \langle F'(z), F(w) \rangle - \frac{\langle F(z), F'(w) \rangle}{|F(z)|^2} \langle F'(z), F(z) \rangle \quad (14)$$

$$+ \frac{\langle F(z), F(w) \rangle \langle F(w), F'(w) \rangle}{|F(z)|^2 |F(w)|^2} \langle F'(z), F(z) \rangle \quad (15)$$

We denote $\frac{\partial}{\partial z} E(z, w)$ and $\frac{\partial^2}{\partial z \partial \bar{w}} E(z, w)$ considered as functions of (z, w) by $E_z(z, w)$ and $E_{z\bar{w}}(z, w)$ respectively. So in particular

$$E_{z\bar{w}}(z, z) = \frac{1}{|F(z)|^2} [\langle F'(z), F'(z) \rangle - \frac{|\langle F'(z), F(z) \rangle|^2}{|F(z)|^2}]$$

Now we calculate $\sin^2(d_N(\Phi_N(w), H_{z \cap w}))$.

First we have

$$O_w(T_\infty(z)) = T_\infty(z) - \frac{\langle T_\infty(z), T_\infty(w) \rangle}{|T_\infty(w)|^2} T_\infty(w)$$

since $O_w(T_\infty(z))$ is orthogonal to $T_\infty(w)$ we have

$$|O_w(T_\infty(z))|^2 = |T_\infty(z)|^2 - \frac{|\langle T_\infty(z), T_\infty(w) \rangle|^2}{|T_\infty(w)|^2}$$

and since $F(w)$ is also orthogonal to $T_\infty(w)$ we also have

$$|\langle F(w), O_w(T_\infty(z)) \rangle|^2 = |\langle F(w), T_\infty(z) \rangle|^2$$

Since

$$\langle T_\infty(z), F(w) \rangle = \langle F'(z), F(w) \rangle - \frac{\langle F'(z), F(z) \rangle}{|F(z)|^2} \langle F(z), F(w) \rangle$$

We get the following equation

Lemma 4.2. *With the notations above*

$$E_z(z, w) = \frac{\langle F(w), F(z) \rangle}{|F(z)|^2 |F(w)|^2} \langle T_\infty(z), F(w) \rangle$$

Moreover,

$$|T_\infty(z)|^2 = \langle F'(z), F'(z) \rangle - \frac{|\langle F'(z), F(z) \rangle|^2}{|F(z)|^2}$$

and

$$\langle T_\infty(z), T_\infty(w) \rangle = \langle F'(z), F'(w) \rangle - \frac{\langle F(w), F'(w) \rangle}{|F(w)|^2} \langle F'(z), F(w) \rangle \quad (16)$$

$$- \frac{\langle F(z), F'(w) \rangle}{|F(z)|^2} \langle F'(z), F(z) \rangle \quad (17)$$

$$+ \frac{\langle F(z), F(w) \rangle \langle F(w), F'(w) \rangle}{|F(z)|^2 |F(w)|^2} \langle F'(z), F(z) \rangle \quad (18)$$

Therefore we have

$$E_{z\bar{w}}(z, z) = \frac{1}{|F(z)|^2} |T_\infty(z)|^2$$

and

$$E_{z\bar{w}}(z, w) = \frac{\langle F(w), F(z) \rangle}{|F(z)|^2 |F(w)|^2} \langle T_\infty(z), T_\infty(w) \rangle$$

Combining these equations we have

$$\sin^2(d_N(\Phi_N(w), H_{z \cap w})) \quad (19)$$

$$= \frac{(|F(z)|^2 |F(w)|^2)^2 |E_z(z, w)|^2}{| \langle F(w), F(z) \rangle |^2 |F(w)|^2 \{ |F(z)|^2 E_{z\bar{w}}(z, z) - \frac{[|F(z)|^2 |F(w)|^2]^2 |E_{z\bar{w}}(z, w)|^2}{|F(w)|^2 E_{z\bar{w}}(w, w) | \langle F(w), F(z) \rangle |^2} \}} \quad (20)$$

$$= \frac{|E_z(z, w)|^2}{E(z, w) [E_{z\bar{w}}(z, z) - \frac{|E_{z\bar{w}}(z, w)|^2}{E(z, w) E_{z\bar{w}}(w, w)}]} \quad (21)$$

So we have the following theorem

Theorem 4.3. *With the notations above we have the equation*

$$\sin^2(d_N(\Phi_N(w), H_{z \cap w})) = \frac{|E_z(z, w)|^2}{E(z, w) E_{z\bar{w}}(z, z) [1 - \frac{|E_{z\bar{w}}(z, w)|^2}{E_{z\bar{w}}(z, z) E_{z\bar{w}}(w, w)} \frac{1}{E(z, w)}]}$$

As in the last section, we choose local coordinates such $z_0 = 0$. Then

$$E(z, w) = P_N^2(z, w) = e^{-|u-v|^2} [1 + R_N(u, v)]^2,$$

where $u = \sqrt{N}z, v = \sqrt{N}w$. So

$$\frac{\partial}{\partial z} E(z, w) = \sqrt{N} \frac{\partial}{\partial u} e^{-|u-v|^2} [1 + R_N(u, v)]^2 \quad (22)$$

$$= \sqrt{N} [e^{-|u-v|^2} (\bar{v} - \bar{u}) [1 + R_N(u, v)]^2] \quad (23)$$

$$+ e^{-|u-v|^2} 2(1 + R_N(u, v)) \frac{\partial}{\partial u} R_N(u, v) \quad (24)$$

When $z = 0$

$$E(0, w) = e^{-N|w|^2} [1 + O(N^{1/2+\varepsilon}|w|^2)]^2$$

$$\frac{\partial}{\partial z} E(0, w) = \sqrt{N} e^{-N|w|^2} (1 + O(N^{-1/2+\varepsilon})) \sqrt{N} \bar{w} + O(N^\varepsilon |w|) \quad (25)$$

$$= (1 + o(1)) N e^{-N|w|^2} \bar{w} \quad (26)$$

where we make $\varepsilon < 1/2$ and use the estimation that

$$|\frac{\partial}{\partial u} R_N(u, v)_{u=0}| \leq C_2 |v| N^{-1/2+\varepsilon}$$

Furthermore we can calculate

$$\frac{\partial^2}{\partial z \partial \bar{w}} E(z, w) = N \frac{\partial}{\partial \bar{v}} \{ \sqrt{N} [e^{-|u-v|^2} (\bar{v} - \bar{u})] [1 + R_N(u, v)]^2 \} \quad (27)$$

$$+ e^{-|u-v|^2} 2(1 + R_N(u, v)) \frac{\partial}{\partial u} R_N(u, v) \} \quad (28)$$

$$= N e^{-|u-v|^2} (u - v) [(\bar{v} - \bar{u})(1 + R_N)^2 + 2(1 + R_N) \frac{\partial}{\partial u} R_N] \quad (29)$$

$$+ N e^{-|u-v|^2} \{ (1 + R_N)^2 + (\bar{v} - \bar{u}) \frac{\partial}{\partial \bar{v}} (1 + R_N)^2 \} \quad (30)$$

$$+ 2 \frac{\partial}{\partial \bar{v}} R_N \frac{\partial}{\partial u} R_N + 2(1 + R_N) \frac{\partial^2}{\partial u \partial \bar{v}} R_N \} \quad (31)$$

Therefore

$$E_{z\bar{w}}(w, w) = N[(1 + R_N)^2 + 2 \frac{\partial}{\partial \bar{v}} R_N \frac{\partial}{\partial u} R_N](v, v) \quad (32)$$

$$= N(1 + 2 \frac{\partial^2}{\partial u \partial \bar{v}} R_N(v, v)) \quad (33)$$

and

$$E_{z\bar{w}}(0, w) = N e^{-N|w|^2} \{ (1 + O(N^{1/2+\varepsilon}|w|^2))^2 (1 - N|w|^2) \} \quad (34)$$

$$+ 2(1 + O(N^{1/2+\varepsilon}|w|^2)) O(N^{1/2+\varepsilon}|w|^2) \quad (35)$$

$$+ O(N^{1/2+\varepsilon}|w|^2) (1 + O(N^{1/2+\varepsilon}|w|^2)) \quad (36)$$

$$+ O(N^{2\varepsilon}|w|^2) + 2(1 + O(N^{1/2+\varepsilon}|w|^2)) \partial u \partial \bar{v} R_N(0, v) \} \quad (37)$$

$$= N e^{-N|w|^2} [1 - (1 + o(1))N|w|^2 + 2 \frac{\partial^2}{\partial u \partial \bar{v}} R_N(0, v)] \quad (38)$$

With the notations above we have the following theorem

Theorem 4.4. *There exist $r(L, h) > 0$, which is independent of N , such that*

$$\sin^2(d_N(\Phi_N(w), H_{0 \cap w})) > r(L, h)$$

for $|w| < \frac{1}{\sqrt{2N}}$

Proof. Before applying theorem 4.3, we need the following estimations

$$\frac{\partial^2}{\partial u \partial \bar{v}} R_N(0, v) - \frac{\partial^2}{\partial u \partial \bar{v}} R_N(0, 0) = A(v) + O(N^{-1/2+\varepsilon}|v|^2) \quad (39)$$

$$\frac{\partial^2}{\partial u \partial \bar{v}} R_N(v, v) - \frac{\partial^2}{\partial u \partial \bar{v}} R_N(0, 0) = A(v) + \overline{A(v)} + O(N^{-1/2+\varepsilon}|v|^2) \quad (40)$$

where $A(v) = O(N^{-1/2+\varepsilon}|v|)$.

Note first that $R_N(u, v)$ is a real analytic function, so we can write $R_N(u, v)$ as power series in (u, \bar{u}, v, \bar{v}) . So the first equation follows directly from theorem [[SZ1]]. Now we prove the second equation.

We denote by $g(u, v)$ the homogeneous part of degree 3 in the power series, since this is the part that contribute terms of degree 1 in the second derivatives. Notice that $g_{u\bar{v}}(0, 0) = 0$, we need to show that

$$g_{u\bar{v}}(0, x) + \overline{g_{u\bar{v}}(0, x)} = g_{u\bar{v}}(x, x)$$

for all $x \in \mathbb{C}$.

Since $g(u, v)$ is real, $\overline{g_{u\bar{v}}(0, x)} = g_{v\bar{u}}(0, x)$. We write $p(x) = g_{u\bar{v}}(x, x) - g_{u\bar{v}}(0, x) - g_{v\bar{u}}(0, x)$. So $p(x)$ is linear in (x, \bar{x}) and $p(0) = 0$. To show that $p(x) \equiv 0$, we just need to show that $\frac{\partial}{\partial x}p(x) = 0$ and $\frac{\partial}{\partial \bar{x}}p(x) = 0$. But

$$\frac{\partial}{\partial x}p(x) = g_{uu\bar{v}}(x, x) + g_{vu\bar{v}}(x, x) - g_{vu\bar{v}}(0, x) - g_{vv\bar{u}}(0, x) = 0$$

where the second equation follows from the fact that all terms in the middle are constant and that g is symmetric with respect to u and v . So we have proved the second equation.

We let $a = \frac{\partial^2}{\partial u \partial \bar{v}}R_N(0, 0)$, then $a = O(N^{-1/2+\varepsilon})$ and a is real. So

$$E_{z\bar{w}}(0, 0) = N(1 + 2a) \quad (41)$$

$$E_{z\bar{w}}(0, w) = Ne^{-N|w|^2}[1 - (1 + o(1))N|w|^2 + 2a + 2A(v) + O(N^{-1/2+\varepsilon}|v|^2)] \quad (42)$$

$$E_{z\bar{w}}(w, w) = N[1 + 2a + 2A(v) + 2\overline{A(v)} + O(N^{-1/2+\varepsilon}|v|^2)] \quad (43)$$

Plug in these estimations together with the ones about $E_z(0, w)$ and $E(0, w)$ to the expression of $\sin^2(d_N(\Phi_N(w), H_{0 \cap w}))$ in the last theorem, and use the Taylor series of $e^{-N|w|^2}$, both the numerator and denominator is bounded by positive multiples of $N|w|^2$, then it is easy to see that there is a constant $r > 0$ independent of N and w for $w < \frac{1}{\sqrt{2N}}$ such that $\sin^2(d_N(\Phi_N(w), H_{0 \cap w})) > r$. Also since the constant in the approximation of the normalized Szegő kernel is independent of the point z , r can be chosen independent of z . \square

As a corollary, we have the following theorem

Theorem 4.5. *Let r_N be the critical radius of $\Phi_N(X)$ considered as a submanifold of \mathbb{CP}^n . There exists a constant $c(X, h) > 0$, such that $r_N > c(X, h)$*

Proof. We still use the preferred coordinates chosen centered at z

Let $N_x(b) = \{v \in N_x(\Phi_N(X)), \|v\| \leq b\}$ Notice that by theorem 2.2 and 2.3, for $w \geq \frac{1}{\sqrt{2N}}$, and for N big enough, $d_N(\Phi_N(z), \Phi_N(z + w)) \geq \cos^{-1}[(1 + o(1))e^{-1/4}]$.

Combining this fact and theorem 4.4, there exists a constant $c > 0$, which is independent of z such that for any point $q \in \Phi_N(X)$,

$$\exp_{\Phi_N(z)}(N_{\Phi_N(z)}(c)) \cap \exp_q(N_q(c)) = \emptyset$$

This implies that the critical radius is bounded below, namely $r_N > c(X, h)$ \square

4.2. Higher Dimension. Actually the argument for Riemann surfaces carries directly to high dimensional Kähler manifolds. Now use the notations in section 3, we have the following theorem

Theorem 4.6. *Let r_N be the critical radius of $\Phi_N(M)$ considered as a submanifold of \mathbb{CP}^n . There exists a constant $c(M, h) > 0$, such that $r_N > c(M, h)$*

Proof. We just need a high dimensional version of theorem 4.4.

We still choose a preferred coordinates centered at z , and let $w < \frac{1}{\sqrt{2N}}$.

In order to apply theorem 4.3, we let X be the complex line in the coordinates chart connecting 0 and w , and restrict Φ_N to an open set $V \subset X$. Then all the estimations we used in proving theorem 4.4 hold for V . So theorem 4.4 can be applied to V . Notice that the lower bounds we can get come from the approximation of the normalized Szegő kernel of $(L, h) \rightarrow M$, hence is independent of w .

Notice that the normal space of M at $\Phi_N(z)$ is contained in the normal hyperplane of $\Phi_N(V)$ at $\Phi_N(z)$, the same is true for $\Phi_N(z + w)$. Therefore the intersection of the two normal spaces $N_{0 \cap w} = N_{\Phi_N(z)}(M) \cap N_{\Phi_N(z+w)}(M)$ of M is contained in the intersection of the two normal hyperplanes. Therefore the distance from $\Phi_N(z)$ to the intersection $N_{0 \cap w}$ is also bounded below independent of z , w and N .

Now use the same argument as in theorem 4.5, we get the expected conclusion. \square

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